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REFLECTION AND TRANSMISSION OF OBLIQUE PLANE WAVES AT A PLANE I--ETC(U)
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Reflection and Transmission of Oblique Plane Waves

at a Plane Interface Between Viscoelastic Media

by

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ABSTRACT

The solution is obtained for the reflection and transmission of time harmonic plane dilatational or shear waves at a plane interface between two linearly viscoelastic materials. Except in special cases, the reflected and transmitted waves are general plane waves, i.e., plane waves whose amplitudes vary across their wave fronts. The angles of reflection and transmission depend on the incident angle in a more complicated way than in the limiting elastic case because the wave speeds of the reflected and transmitted waves are, in general, functions of the incident angle. Necessary and sufficient conditions for the existence of interface waves are obtained. It is found that no interface waves can exist for some materials. For another class of materials, interface waves exist for discrete angles of incidence. For still other materials, interface waves exist for all incident angles greater than some critical angle.

INTRODUCTION

This paper treats the reflection and transmission of a time harmonic plane dilatational or shear wave by a plane interface, $y = 0$, between two homogeneous and isotropic linearly viscoelastic materials as indicated by Figure 1. The method of solution is the same as the procedure used in solving the elastic problem by use of potential functions except that it leads to the definition of complex reflection and transmission angles. Further analysis of the formal solution to obtain the real reflection and transmission angles leads to the definition of general plane waves, i.e., plane waves whose amplitudes vary across their wave fronts. This result is consistent with previous work[1] dealing with the reflection of plane waves from plane rigid and free boundaries where such waves were defined initially so that no complex angles would enter into the analysis. That procedure could also be applied to the interface problem; however, the method given here leads more directly to a formal representation of the solution.

The problems treated here and in Reference 1 have been considered by Lockett[2] by using a method which is different than either of the approaches mentioned above. Our results agree in general with his, except that we present a bit more detailed analysis of the reflection and refraction phenomena and obtain a few more explicit results. In particular, we obtain necessary and sufficient conditions for the existence of interface waves and find that, in general, such waves can exist only for distinct incident angles, a result which apparently was not recognized from his analysis. We also obtain explicit relationships for the transmitted and reflected wave speeds and attenuation coefficients.

As might be expected from the previous work, the phase velocities of the reflected and transmitted waves are functions of the incident angle as well as material properties, and the real reflection and transmission angles depend on the incident angle in a more complicated way than in the elastic case. Also, as indicated above, interface waves are generally possible only for discrete incident angles; whereas, in the elastic case or in a special viscoelastic case, they are possible for a range of incident angles. In the elastic case this phenomena is referred to as total reflection or refraction[3]. In general, a phase shift between the incident and the reflected or refracted waves is introduced by the interface.

In Section I, the problem is formulated and, in Section II the formal solution, for either an incident dilatational or shear wave, is obtained. An analysis and interpretation of the physical consequences of the formal solution is presented in Section III. Finally the results are summarized in Section IV.

I. FORMULATION

Let $\underset{\sim}{u}^0(\underset{\sim}{x}, t) = \underset{\sim}{u}(\underset{\sim}{x}) \exp\{-i\omega t\}$ and $\tau^0_{ij}(\underset{\sim}{x}, t) = \tau_{ij}(\underset{\sim}{x}) \exp\{-i\omega t\}$ denote the time harmonic particle displacement vector and stress tensor where $\omega > 0$ is the frequency, t is time, and $\underset{\sim}{x} = (x_1, x_2, x_3)$ is a position vector in Cartesian coordinates. Here, the subscripts i, j take on the values 1, 2, 3. Hereafter, the complex quantities $\underset{\sim}{u}$ and τ_{ij} will be referred to as the particle displacement and stress tensor. The equations of motion, with body forces neglected, for small amplitude time harmonic waves in a homogeneous and isotropic linearly viscoelastic material are[4]

$$\mu\Delta\underset{\sim}{u} + (\lambda + \mu)\nabla(\nabla \cdot \underset{\sim}{u}) + \rho\omega^2\underset{\sim}{u} = 0 \qquad (1)$$

where $\lambda(\omega)$ and $\mu(\omega)$ are specified complex valued material properties which reduce to the Lamé constants in the limiting elastic case. The constant density is denoted by ρ, and the gradient and Laplace operators are denoted by ∇ and Δ. The constitutive relations are

$$\tau_{ij} = \lambda\delta_{ij}\, \nabla\cdot\underset{\sim}{u} + \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \qquad (2)$$

where δ_{ij} is the Kronecker delta and i, j = 1, 2, 3.

We consider two-dimensional motions of the form $\underset{\sim}{u} = \underset{\sim}{u}(x, y, 0)$ where $x_1 = x$, $x_2 = y$, $x_3 = z$ are the usual Cartesian coordinates. Since $\underset{\sim}{u}$ is independent of z, u_3 is uncoupled from u_1 and u_2 in Eq. 1 and u_3 satisfies the z component of Eq. 1. The solutions of this equation describe shear motions transverse to the x, y plane. They have been previously investigated for a restricted class of viscoelastic materials[5] and are not considered here.

Only the "in plane" motions described by the x and y components of Eq. 1 are studied. Hence, we take $u_3 = 0$, $u_1 = u$ and $u_2 = v$ in Eqs. 1 and 2.

In the following analysis, the subscript $m = 1, 2$ is used to indicate which medium the subscripted variable refers to, i.e., $m = 1$ implies $y > 0$ and $m = 2$ implies $y < 0$, as shown in Fig. 1. The subscript $n = 1, 2$ is used to denote dilatational and shear waves respectively, and the subscript $\ell = 1, 2$ denotes the case where a dilatational ($\ell = 1$) or a shear ($\ell = 2$) wave is incident at the interface.

Eight potential functions, $\Phi_{\ell mn}$, are defined such that, for $\ell, m = 1, 2$,

$$u_{\ell m} = \partial\Phi_{\ell m1}/\partial x + \partial\Phi_{\ell m2}/\partial y,$$

$$v_{\ell m} = \partial\Phi_{\ell m1}/\partial y - \partial\Phi_{\ell m2}/\partial x, \tag{3}$$

are the displacement components. The displacement vector obtained from Eq. 3 is a solution of Eq. 1 provided

$$\Delta\Phi_{\ell mn} + k_{mn}^2\,\Phi_{\ell mn} = 0, \qquad \ell, m, n = 1, 2, \tag{4}$$

where

$$k_{mn} \equiv \omega/S_{mn};$$

$$S_{m1}^2 = (\lambda_m + 2\mu_m)/\rho_m;\ S_{m2}^2 = \mu_m/\rho_m. \tag{5}$$

Here, k_{mn} is the "complex wave number" for dilatational ($n = 1$) or shear ($n = 2$) waves in medium m which may be expressed as[1]

$$k_{mn} = (\omega/C_{mn})(1 + i \tan \Omega_{mn}), \tag{6}$$

where

$$\begin{aligned} &\tan 2\Omega_{mn} \equiv - mS^2_{mn}/R_\ell S^2_{mn}, \ 0 \leq \Omega_{mn} < \pi/2; \\ &C_{mn} \equiv |S_{mn}| \sec \Omega_{mn}, \end{aligned} \tag{7}$$

for m, n = 1, 2. The complex quantities, S_{mn}, are referred to as "complex wave speeds" because of their analogy to the elastic wave speeds. It will be shown that the real quantities, C_{mn}, are least upper bounds for the dilatational (n = 1) and shear (n = 2) wave speeds in medium m. The requirement that $0 \leq \Omega_{mn} < \pi/2$ comes from the condition that plane waves do not grow in amplitude as they propagate[1].

An effort has been made to keep the notation consistant with that in Ref. 1. However, in order to avoid confusion with the Kronecker delta, Ω is used for the variable defined as δ in the previous work. Also, we have reserved ϕ to denote an angle. Hence, the potential functions are here denoted by Φ.

The normal stresses in the x and y directions, for medium m and incident wave ℓ, are denoted by $\sigma_{x\ell m}$ and $\sigma_{y\ell m}$, and the corresponding shear stress is denoted by $\tau_{\ell m}$. Thus, from Eqs. 2 and 3, for ℓ, m = 1, 2,

$$\begin{aligned} \sigma_{x\ell m} &= 2\mu_m(\partial^2\Phi_{\ell m1}/\partial x^2 + \partial^2\Phi_{\ell m2}/\partial x\partial y) - \lambda_m k^2_{m1} \Phi_{\ell m1}, \\ \sigma_{y\ell m} &= 2\mu_m(\partial^2\Phi_{\ell m1}/\partial y^2 - \partial^2\Phi_{\ell m2}/\partial x\partial y) - \lambda_m k^2_{m1} \Phi_{\ell m1}, \\ \tau_{\ell m} &= \mu_m(2\partial^2\Phi_{\ell m1}/\partial x\partial y + \partial^2\Phi_{\ell m2}/\partial y^2 - \partial^2\Phi_{\ell m2}/\partial x^2). \end{aligned} \tag{8}$$

The boundary conditions at the interface, $y = 0$, are, for $\ell = 1, 2$,

$$u_{\ell 1}(x, 0) = u_{\ell 2}(x, 0), \qquad v_{\ell 1}(x, 0) = v_{\ell 2}(x, 0),$$
$$\sigma_{y\ell 1}(x, 0) = \sigma_{y\ell 2}(x, 0), \qquad \tau_{\ell 1}(x, 0) = \tau_{\ell 2}(x, 0). \tag{9}$$

The total potential functions are defined, for $\ell, n = 1, 2$, by

$$\phi_{\ell 1n} = \delta_{\ell n}\, \Psi_{\ell} + \psi_{\ell 1n};\ \phi_{\ell 2n} = \psi_{\ell 2n} \tag{10}$$

where $\delta_{\ell n}$ is the Kronecker delta. The incident wave, Ψ_{ℓ}, is a dilatational wave if $\ell = 1$ and a shear wave if $\ell = 2$. The reflected and refracted waves are denoted by $\psi_{\ell mn}$. The potential functions for $\ell, m, n = 1, 2$ are assumed to be of the form

$$\Psi_{\ell} = (I_{\ell} S_{\ell 1}/\omega) \exp\{ik_{1\ell}(\underset{\sim}{r}_{\ell} \cdot \underset{\sim}{x})\}\ ,$$
$$\psi_{\ell mn} = (I_{\ell} R_{\ell mn} S_{mn}/\omega) \exp\{ik_{mn}(r_{\ell mn} \cdot \underset{\sim}{x})\}\ , \tag{11}$$

where

$$\underset{\sim}{r}_{\ell} \equiv \underset{\sim}{i} \sin\theta_{\ell} - \underset{\sim}{j} \cos\theta_{\ell}, \quad \underset{\sim}{x} \equiv \underset{\sim}{i}x + \underset{\sim}{j}y,$$
$$\underset{\sim}{r}_{\ell mn} \equiv \underset{\sim}{i} \sin\zeta_{\ell mn} + \underset{\sim}{j}\varepsilon_{m} \cos\zeta_{\ell mn},\ \varepsilon_{m} \equiv (-1)^{m+1}. \tag{12}$$

Here, $\underset{\sim}{i}$, $\underset{\sim}{j}$ are unit vectors along the positive x and y axes. The angle, θ_{ℓ}, is the angle between the incident dilatational ($\ell = 1$) or shear ($\ell = 2$) ray $\underset{\sim}{r}_{\ell}$ and $\underset{\sim}{j}$. The angles $\zeta_{\ell mn}$ are, in general, complex. If they are real, then they are between $\underset{\sim}{r}_{\ell mn}$ and $\underset{\sim}{j}$ for $m = 1$ and between $\underset{\sim}{r}_{\ell mn}$ and $-\underset{\sim}{j}$ for $m = 2$ (and are given by $\zeta_{\ell mn} = \theta_{\ell mn}$). See Fig. 1.

It may be shown that I_ℓ is the displacement amplitude of the incident wave, normal to its wavefront, at the origin $x = y = 0$. Similarly, $R_{\ell mn} I_\ell$ is the displacement amplitude of the reflected or refracted wave, normal to its wavefront, at the origin. Thus, $R_{\ell mn}$, m, n = 1, 2, are the displacement (or velocity) reflection and refraction coefficients for $\ell = 1, 2$.

Note that the incident wave is a plane wave in the usual sense, i.e., it has constant amplitude across its wavefront. When $\zeta_{\ell mn}$ is complex, it will be shown that the reflected or refracted wave is a "general plane wave", i.e., it is attenuated across its wavefront.

From Eqs. 3, 4, 11 and 12 it may be shown that

$$\begin{bmatrix} u_{\ell m} \\ v_{\ell m} \\ \sigma_{x\ell m} \\ \sigma_{y\ell m} \\ \tau_{\ell m} \end{bmatrix} = \begin{bmatrix} ik_{m1} \sin \zeta_{\ell m1} & ik_{m2}\, \varepsilon_m \cos \zeta_{\ell m2} \\ \varepsilon_m ik_{m1} \cos \zeta_{\ell m1} & -ik_{m2} \sin \zeta_{\ell m2} \\ -k_{m1}^2(\lambda_m + 2\mu_m \sin^2 \zeta_{\ell m1}) & -\mu_m k_{m2}^2 \varepsilon_m \sin 2\zeta_{\ell m2} \\ -k_{m1}^2(\lambda_m + 2\mu_m \cos^2 \zeta_{\ell m1}) & \mu_m k_{m2}^2 \varepsilon_m \sin 2\zeta_{\ell m2} \\ -\varepsilon_m \mu_m k_{m1}^2 \sin 2\zeta_{\ell m1} & -\mu_m k_{m2}^2 \cos 2\zeta_{\ell m2} \end{bmatrix} \begin{bmatrix} \psi_{\ell m1} \\ \\ \psi_{\ell m2} \end{bmatrix}$$

$$+ \delta_{m1} \Psi_\ell \begin{bmatrix} i(\delta_{\ell 1} k_{11} \sin \theta_1 - \delta_{\ell 2} k_{12} \cos \theta_2) \\ -i(\delta_{\ell 1} k_{11} \cos \theta_1 + \delta_{\ell 2} k_{12} \sin \theta_2) \\ -\delta_{\ell 1} k_{11}^2(\lambda_1 + 2\mu_1 \sin^2 \theta_1) + \delta_{\ell 2}\mu_1 k_{12}^2 \sin 2\theta_2 \\ -\delta_{\ell 1} k_{11}^2(\lambda_1 + 2\mu_1 \cos^2 \theta_1) - \delta_{\ell 2}\mu_1 k_{12}^2 \sin 2\theta_2 \\ \delta_{\ell 1}\mu_1 k_{11}^2 \sin 2\theta_1 - \delta_{\ell 2}\mu_1 k_{12}^2 \cos 2\theta_2 \end{bmatrix} \qquad (13)$$

II. SOLUTION

The boundary conditions given by Eq. 9 require that all of the exponentials in Eq. 13 be equal on $y = 0$. Hence, we obtain the complex Snell's law, for ℓ, m, n = 1, 2,

$$\sin \zeta_{\ell mn} = S_{mn} \sin \theta_\ell / S_{1\ell} , \tag{14}$$

where the complex wave speeds, S_{mn}, are defined by Eq. 5. Rearranging the expressions resulting from the boundary conditions and combining the result with Eq. 14, we obtain the linear set of equations

$$\underline{\underline{A}}_\ell \, \underset{\sim}{R}_\ell = \underset{\sim}{B}_\ell \ , \ \ell = 1, 2, \tag{15}$$

where

$$\underline{\underline{A}}_\ell = \begin{bmatrix} \sin \zeta_{\ell 11} & \cos \zeta_{\ell 12} & -\sin \zeta_{\ell 21} & \cos \zeta_{\ell 22} \\ \cos \zeta_{\ell 11} & -\sin \zeta_{\ell 12} & \cos \zeta_{\ell 21} & \sin \zeta_{\ell 22} \\ -\rho_1 S_{11} \cos \zeta_{\ell 12} & \rho_1 S_{12} \sin 2\zeta_{\ell 12} & \rho_2 S_{21} \cos 2\zeta_{\ell 22} & \rho_2 S_{22} \sin 2\zeta_{\ell 22} \\ \dfrac{\rho_1 S_{12}^2 \sin 2\zeta_{\ell 11}}{S_{11}} & \rho_1 S_{12} \cos 2\zeta_{\ell 12} & \dfrac{\rho_2 S_{22}^2 \sin 2\zeta_{\ell 21}}{S_{21}} & -\rho_2 S_{22} \cos 2\zeta_{\ell 22} \end{bmatrix} \tag{16}$$

$$\underset{\sim}{B}_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ \rho_1 S_{11} \cos 2\zeta_{112} \\ \rho_1 S_{12}^2 \sin 2\theta_1 \end{bmatrix}, \quad \underset{\sim}{B}_2 = \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ -\rho_1 S_{12} \sin 2\theta_2 \\ -\rho_1 S_{12} \cos 2\theta_2 \end{bmatrix}, \quad \underset{\sim}{R}_\ell = \begin{bmatrix} R_{\ell 11} \\ R_{\ell 12} \\ R_{\ell 21} \\ R_{\ell 22} \end{bmatrix}. \tag{17}$$

If det $|\underline{A}_\ell| \neq 0$, then the reflection and refraction coefficients, $R_{\ell mn}$, are determined by inverting Eq. 15. In general, $R_{\ell mn}$ is complex. Hence, there is a phase shift between the incident wave and the reflected and refracted waves in general. In order for the formal solution obtained above to be unique, unique values for $\zeta_{\ell mn}$, $\ell,m,n = 1,2$, must be specified (since the inversion of Eq. 14 yields multivalued roots for the complex angles, $\zeta_{\ell mn}$). In the following section, the physics of the reflection and reflection phenomena will be studied, and the information presented there will allow a unique inversion of Eq. 14.

III. PHYSICAL INTERPRETATION OF THE SOLUTION

Let the complex angle, $\zeta_{\ell mn}$, be represented by

$$\zeta_{\ell mn} = \alpha_{\ell mn} + i\beta_{\ell mn} \tag{18}$$

where $\alpha_{\ell mn}$ and $\beta_{\ell mn}$ real numbers. Combining Eqs. 5, 6, 14, and 18; noting that $\sin \zeta_{\ell mn} = \sin \alpha_{\ell mn} \cosh \beta_{\ell mn} + i \cos \alpha_{\ell mn} \sinh \beta_{\ell mn}$; and equating the real and imaginary parts of the resulting expression, we obtain

$$\sin \alpha_{\ell mn} \cosh \beta_{\ell mn} = \Gamma_{\ell mn} \cos \Delta_{\ell mn} , \tag{19}$$

$$\cos \alpha_{\ell mn} \sinh \beta_{\ell mn} = \Gamma_{\ell mn} \sin \Delta_{\ell mn} \tag{20}$$

where

$$\Gamma_{\ell mn} \equiv \gamma_{\ell mn} \cos \Omega_{mn} / \cos \Omega_{1\ell} \geq 0, \tag{21}$$

$$\gamma_{\ell mn} \equiv C_{mn} \sin \theta_{\ell} / C_{1\ell} \geq 0 , \tag{22}$$

$$\Delta_{\ell mn} \equiv \Omega_{1\ell} - \Omega_{mn} . \tag{23}$$

Combining Eqs. 19 and 20 and using simple identities, we obtain

$$\sinh^2 \beta_{\ell mn} + \sin^2 \alpha_{\ell mn} = \Gamma^2_{\ell mn} , \tag{24}$$

$$\cosh^2 \beta_{\ell mn} - \cos^2 \alpha_{\ell mn} = \Gamma^2_{\ell mn} , \tag{25}$$

$$\sin \alpha_{\ell mn} \cosh\beta_{\ell mn} \sin \Delta_{\ell mn} = \cos \alpha_{\ell mn} \sinh \beta_{\ell mn} \cos \Delta_{\ell mn} . \tag{26}$$

Combining Eq. 24 with the square of Eq. 26 and noting that $\cosh^2\beta_{\ell mn} - \sinh^2\beta_{\ell mn} = 1$, we obtain a quadratic in $\sinh^2\beta_{\ell mn}$ whose solution for $\sinh^2\beta_{\ell mn} \geq 0$ is

$$\sinh^2\beta_{\ell mn} = \frac{1}{2}\left\{\Gamma^2_{\ell mn} - 1 + \left[(1 - \Gamma^2_{\ell mn})^2 + 4\Gamma^2_{\ell mn}\sin^2\Delta_{\ell mn}\right]^{\frac{1}{2}}\right\} \tag{27}$$

Note that $\zeta_{\ell mn}$ is real if and only if $\beta_{\ell mn} = 0$ in Eq. 18. Thus, from Eq. 27, <u>necessary and sufficient conditions for $\zeta_{\ell mn}$ to be a real, non-zero angle are $\Delta_{\ell mn} = 0$ and $0 < \Gamma_{\ell mn} \leq 1$.</u> In this case, $\zeta_{\ell mn} = \alpha_{\ell mn} = \theta_{\ell mn}$ is given by Eq. 14 or Eq. 19. Note that $\Gamma_{\ell mn} = 0$ ($\theta_\ell = 0$) is a sufficient condition for $\beta_{\ell mn} = 0$, and in this case $\alpha_{\ell mn} = 0$ from Eq. 19. Hence, if the incident wave is normal to the interface, all reflected and refracted waves propagate in directions normal to the interface.

If $\beta_{\ell mn} \neq 0$, then the proper sign for $\sinh\beta_{\ell mn} = |\sinh\beta_{\ell mn}|$ given by Eq. 27 must be determined before the solution is complete. Since $\Gamma_{\ell mn} \geq 0$, Eq. 19 indicates that (note that $0 \leq \Omega_{mn} < \pi/2$, so $-\pi/2 < \Delta_{\ell mn} < \pi/2$)

$$0 \leq \alpha_{\ell mn} \leq \pi \tag{28}$$

Eq. 20 then indicates only that $\cos\alpha_{\ell mn}$ and $\beta_{\ell mn}$ either have the same or the opposite sign depending on the sign of $\Delta_{\ell mn}$. Thus, the form of the information presented so far is not sufficient to determine the proper sign for $\beta_{\ell mn}$. We shall obtain the necessary information by considering the real properties of the reflection phenomena.

Combining Eqs. 11, 12, and 18 and simplifying, we obtain

$$ik_{mn}\underset{\sim}{r}_{\ell mn} = (i\omega/c_{\ell mn})\ (\underset{\sim}{p}_{\ell mn} + i\ D_{\ell mn}\underset{\sim}{a}_{\ell mn}) \tag{29}$$

where

$$c_{\ell mn} \equiv C_{mn}\ \xi_{\ell mn}\ , \tag{30}$$

$$\xi_{\ell mn} \equiv \left[1 + \sinh^2 \beta_{\ell mn} \sec^2 \Omega_{mn}\right]^{-\frac{1}{2}}\ , \tag{31}$$

$$\underset{\sim}{p}_{\ell mn} \equiv \underset{\sim}{i} \sin \theta_{\ell mn} + \underset{\sim}{j}\varepsilon_m \cos \theta_{\ell mn}\ , \tag{32}$$

$$\underset{\sim}{a}_{\ell mn} \equiv \underset{\sim}{i} \sin (\alpha_{\ell mn} + \eta_{\ell mn}) + \underset{\sim}{j}\varepsilon_m \cos(\alpha_{\ell mn} + \eta_{\ell mn})\ , \tag{33}$$

$$\theta_{\ell mn} \equiv \alpha_{\ell mn} - \phi_{\ell mn}\ , \tag{34}$$

$$\tan \phi_{\ell mn} = \tan \Omega_{mn} \tanh \beta_{\ell mn}\ ,\ -\pi/2 \leq \phi_{\ell mn} \leq \pi/2\ , \tag{35}$$

$$\tan \eta_{\ell mn} = \cot \Omega_{mn} \tanh \beta_{\ell mn}\ ,\ -\pi/2 \leq \eta_{\ mn} \leq \pi/2, \tag{36}$$

$$D_{\ell mn} \equiv \xi_{\ell mn} \left[\tan^2 \Omega_{mn} \cosh^2 \beta_{\ell mn} + \sinh^2 \beta_{\ell mn}\right]^{\frac{1}{2}}\ . \tag{37}$$

In Eq. 29, $c_{\ell mn}$ is the real wave speed, $\underset{\sim}{p}_{\ell mn}$ and $\underset{\sim}{a}_{\ell mn}$ are real unit vectors in the directions of propagation and damping, and $D_{\ell mn}$ is the total damping coefficient. From Eq. 30, $\xi_{\ell mn}$ is the ratio of the wave speed to the material property C_{mn} given by Eq. 7, and from Eq. 31 $0 < \xi_{\ell mn} \leq 1$. Hence, the material properties C_{mn} are least upper bounds on the wave speeds. Note, from Eq. 31, that $\xi_{\ell mn} = 1$ if and only if $\beta_{\ell mn} = 0$. Thus, a necessary and sufficient condition for waves to propagate with speeds specified entirely by material properties, i.e. independent of the incident angle, is for $\zeta_{\ell mn}$ to be real. Note that this is always the case at normal incidence since $\zeta_{\ell mn} = 0$ in that case.

The propagation and damping unit vectors, $\underset{\sim}{p}_{\ell mn}$ and $\underset{\sim}{a}_{\ell mn}$ are represented schematically in Figure 2 for m = 1. Note that they coincide if and only if $\phi_{\ell mn} = \eta_{\ell mn}$ which occurs if and only if $\beta_{\ell mn} = 0$. Thus, a necessary and sufficient condition for the surfaces of constant phase, $\underset{\sim}{p}_{\ell mn} \cdot \underset{\sim}{x} =$ constant, and surfaces of constant amplitude, $\underset{\sim}{a}_{\ell mn} \cdot \underset{\sim}{x} =$ constant, to be parallel is for $\zeta_{\ell mn}$ to be real. Otherwise, the waves are attenuated in directions other than the direction of propagation.

The real reflection or refraction angle, $\theta_{\ell mn}$, will now be determined. Combining Eqs. 34, 35, and 31 to obtain an expression for $\sin \theta_{\ell mn}$, and then comparing the result with the sum of Eq. 19 and $\tan \Omega_{mn}$ times Eq. 23, we find

$$\sin \theta_{\ell mn} = \xi_{\ell mn} \gamma_{\ell mn} \geq 0. \quad (38)$$

This result, which explicitly determines the reflection and refraction angles once $\xi_{\ell mn}$ is found from Eqs. 31 and 27, is precisely the same form obtained for the reflection of plane waves from free and rigid boundaries[1]. Note that $0 \leq \xi_{\ell mn} \gamma_{\ell mn} \leq 1$ since the analysis has required that $\theta_{\ell mn}$ be real.

The attenuation vector, $\underset{\sim}{a}_{\ell mn}$, may be decomposed into components along and normal to $\underset{\sim}{p}_{\ell mn}$ by noting that $\underset{\sim}{a}_{\ell mn} = (\underset{\sim}{a}_{\ell mn} \cdot \underset{\sim}{p}_{\ell mn})\, \underset{\sim}{p}_{\ell mn} + (\underset{\sim}{a}_{\ell mn} \cdot \underset{\sim}{t}_{\ell mn})\, \underset{\sim}{t}_{\ell mn}$ where $\underset{\sim}{t}_{\ell mn} = \underset{\sim}{k} \times \underset{\sim}{p}_{\ell mn}$ and $\underset{\sim}{k}$ is a unit vector in the $x_3 = z$ direction. Hence, the unit vector tangent to the reflected or refracted wave front, positive when pointing away from the interface, is

$$\underset{\sim}{t}_{\ell mn} = -\underset{\sim}{i} \cos \theta_{\ell mn} + \underset{\sim}{j} \epsilon_m \sin \theta_{\ell mn} \ . \tag{39}$$

Hence, Eq. 29 may be transformed to

$$ik_{mn}(\underset{\sim}{r}_{\ell mn} \cdot \underset{\sim}{x}) = (i\omega/c_{\ell mn}) \Big[(1 + i\, \xi^2_{\ell mn} \tan\Omega_{mn})\, (\underset{\sim}{p}_{\ell mn} \cdot \underset{\sim}{x}) - i\xi^2_{\ell mn} \sec^2 \Omega_{mn} \sinh \beta_{\ell mn} \mathrm{cooh}\, \beta_{\ell mn}\, (\underset{\sim}{t}_{\ell mn} \cdot \underset{\sim}{x}) \Big] \tag{40}$$

Hence, the longitudional damping coefficient (in the direction of $\underset{\sim}{p}_{\ell mn}$) is

$$d_{\ell mn} \equiv (\omega/C_{mn})\, \xi_{\ell mn} \tan \Omega_{mn} \ , \tag{41}$$

and the transverse damping coefficient (in the direction of $\underset{\sim}{t}_{\ell mn}$) is

$$T_{\ell mn} \equiv -\, (\omega/C_{mn})\, \xi_{\ell mn} \sec^2 \Omega_{mn} \sinh \beta_{\ell mn} \cosh \beta_{\ell mn} \ . \tag{42}$$

Thus, the wave is transversely damped away from the interface if $\beta_{\ell mn} < 0$ and toward the interface if $\beta_{\ell mn} > 0$. Note also that $T_{\ell mn} = 0$ if and only if $\beta_{\ell mn} = 0$. Thus, the significance of the sign of $\sinh \beta_{\ell mn}$ now becomes clearer.

Multiplying Eq. 19 by tan Ω_{mn}, adding the result to Eq. 20, simplifying and noting Eqs. 34, 35, 38, 31 and 42, we obtain

$$\omega\gamma_{\ell mn} \, [\xi^2_{\ell mn} \tan \Omega_{mn} - \tan \Omega_{1\ell}\,] = C_{mn} \, T_{\ell mn} \cos \theta_{\ell mn} \tag{43}$$

We note that Eq. 43 is consistent with the previous findings on necessary and sufficient conditions for $\beta_{\ell mn} = 0$ (and, therefore, $T_{\ell mn} = 0$) in that those conditions are included in the set of conditions for which the left side of Eq. 43 vanishes. From the physical situation, $|\theta_{\ell mn}| \le \pi/2$. Thus Eq. 28 indicates that $0 \le \theta_{\ell mn} \le \pi/2$ so that cos $\theta_{\ell mn}$ is non-negative. Hence, the signs of $T_{\ell mn}$ and $\beta_{\ell mn} \neq 0$ are determined by the left side of Eq. 43 provided both sides are not identically zero. Excluding, for the moment, interface waves ($\theta_{\ell mn} = \pi/2$) and the case of no transverse damping,

$$\text{sign } T_{\ell mn} = \text{sign } (-\beta_{\ell mn}) = \text{sign } (\xi^2_{\ell mn} \tan \Omega_{mn} - \tan \Omega_{1\ell});$$

$$\text{if} \qquad \beta_{\ell mn} \neq 0, \ \theta_{\ell mn} \neq \pi/2. \tag{44}$$

Note that if $\Omega_{1\ell} > \Omega_{mn}$, then $T_{\ell mn} < 0$ since $\xi_{\ell mn} \le 1$ and the reflected or transmitted wave is transversely damped toward the interface.

We now determine necessary and sufficient conditions for interface waves to exist, i.e., $\theta_{\ell mn} = \pi/2$. To facilitate the argument, we consider three cases: a) $\Omega_{mn} \neq \Omega_{1\ell}$, b) $\Omega_{mn} = \Omega_{1\ell} \neq 0$, and c) $\Omega_{mn} = \Omega_{1\ell} = 0$.

If $\Omega_{mn} \neq \Omega_{1\ell}$, then from Eq. 28 $\beta_{\ell mn} \neq 0$. Furthermore, we have already shown only normally reflected and transmitted waves exist if $\gamma_{\ell mn} = 0$ (normal incidence). Hence, from Eq. 43, <u>a necessary and</u>

sufficient condition for $\theta_{\ell mn} = \pi/2$ (if $\Omega_{mn} \neq \Omega_{1\ell}$) is

$$\xi^2_{\ell mn} \tan \Omega_{mn} = \tan \Omega_{1\ell} \quad . \tag{45}$$

Note that this condition can not be satisfied if either $\Omega_{mn} = 0$ or $\Omega_{1\ell} = 0$, or if $\Omega_{1\ell} > \Omega_{mn}$ (since $\xi \leq 1$). Thus, no interface wave can exist in these cases. Hence, if $\Omega_{mn} \neq 0$ and $\Omega_{1\ell} \neq 0$, and if $\Omega_{mn} \neq \Omega_{1\ell}$, then from Eq. 38, an interface wave exists if and only if

$$\gamma^2_{\ell mn} = \tan \Omega_{mn}/\tan \Omega_{1\ell} \leq 1, \ \Omega_{mn} \neq \Omega_{1\ell} \ . \tag{46}$$

Note that if an interface wave occurs, then it occurs for a discrete incident angle θ_{ℓ}.

If $\Omega_{mn} = \Omega_{1\ell} \neq 0$, then $\Delta_{\ell mn} = 0$ and we have previously found that $\beta_{\ell mn} = 0$ if and only if $\Gamma_{\ell mn} = \gamma_{\ell mn} \leq 1$. (See Eq. 28.) Thus, from Eq. 31, $\xi_{\ell mn} = 1$ in this case. Hence, from Eq. 38, an interface wave is possible if and only if $\gamma_{\ell mn} \geq 1$. In fact, from Eqs. 43 and 38, a necessary and sufficient condition for $\theta_{\ell mn} = \pi/2$ (if $\Omega_{mn} = \Omega_{1\ell} \neq 0$) is

$$\gamma_{\ell mn} = 1. \tag{47}$$

To make this clear, note that the left side of Eq. 43, in this case reduces to $\gamma_{\ell mn} (1-\xi^2_{\ell mn}) \tan \Omega_{mn}$, which can vanish if and only if $\xi_{\ell mn} = 1$. Thus, from Eq. 38, $\theta_{\ell mn} = \pi/2$ if and only if $\gamma_{\ell mn} = 1$. Note that this implies that an interface wave can exist for only a distinct incident angle.

If $\Omega_{mn} = \Omega_{1\ell} = 0$, then $\beta_{\ell mn} \neq 0$ if $\gamma_{\ell mn} > 1$ as in the previous case. Further, the left side of Eq. 43 is identically zero. Thus, if $\beta_{\ell mn} \neq 0$, then $\theta_{\ell mn} = \pi/2$. Note that $\xi_{\ell mn} = 1$ for $\gamma_{\ell mn} \leq 1$. Thus, from Eq. 38, $\theta_{\ell mn} < \pi/2$ for $\gamma_{\ell mn} < 1$, and $\theta_{\ell mn} = \pi/2$ if $\gamma_{\ell mn} = 1$. Hence, $\theta_{\ell mn} = \pi/2$ for $\gamma_{\ell mn} \geq 1$. For the reciprocal argument, note that if $\theta_{\ell mn} = \pi/2$, then from eq. 38, $\xi_{\ell mn}\, \gamma_{\ell mn} = 1$. Then since $0 < \xi_{\ell mn} \leq 1$, it is clear that $\gamma_{\ell mn} \geq 1$. Thus, <u>a necessary and sufficient condition for $\theta_{\ell mn} = \pi/2$</u> (if $\Omega_{mn} = \Omega_{1\ell} = 0$) is

$$\gamma_{\ell mn} \geq 1. \tag{48}$$

This implies that an interface wave exists for any incident angle such that $\sin \theta_{\ell} > C_{1\ell}/C_{mn}$.

It remains to specify the proper sign for $T_{\ell mn}$ and $\beta_{\ell mn}$ when $\theta_{\ell mn} = \pi/2$. In all cases where interface waves exist, we choose

$$T_{\ell mn} \geq 0,\ \beta_{\ell mn} \leq 0 \text{ for } \theta_{\ell mn} = \pi/2, \tag{49}$$

so that the solution is bounded for increasing y. This choice also provides consistency with the limiting elastic case. Note that if $\Omega_{mn} = \Omega_{1\ell} \neq 0$, then from Eqs. 47 and 28, $\beta_{\ell mn} = 0$.

IV. DISCUSSION

From the preceding analysis, we can make several general observations about the reflection and transmission phenomena at the interface. In general, the reflected and transmitted waves are general plane waves (both longitudinal and transverse damping) whose wave speeds and attenuation coefficients are functions of the incident angle as well as material properties. The least upper bound for the wave speed is the material property C_{mn} where m, n = 1,2. In general, interface waves exist for distinct angles of incidence if they are possible at all. In general, there is a phase shift between the incident and reflected or transmitted wave.

The exceptions to the above general statements center around special cases for the complex wave speeds S_{mn}. If $ImS_{mn} = ImS_{1\ell}$, then the reflected or transmitted wave, $\Phi_{\ell mn}$, is a plane wave in the usual sense (constant amplitude across the wave front) unless it is an interface wave, in which case it becomes a general plane wave. If $ImS_{mn} = ImS_{1\ell} \neq 0$, an interface wave can exist only for distinct incident angles; but if $ImS_{mn} = ImS_{1\ell} = 0$, then interface waves exist for any incident angle greater than some critical angle. Also, the only case where the wave speed is independent of the incident angle is when $ImS_{mn} = ImS_{1\ell}$.

Another special case of some interest is $\Omega_{1\ell} > \Omega_{mn}$. In this case, no interface wave is possible. Also, it was shown that the reflected or transmitted wave is attenuated across its front toward the interface in this case. This result is contrary to the elastic case of internal reflection where the wave is attenuated away from the interface. For the reflected waves ($m = 1$), Lockett[2] has suggested that when the incident wave is a shear wave $\Omega_{11} \geq \Omega_{1n}$ for most real materials. The transmitted wave properties cannot be compared without specifying specific materials. Hence, no such relationship between $\Omega_{1\ell}$ and Ω_{2n} is generally applicable.

The results presented here are entirely consistent with those for rigid and free boundaries in Reference 1. In generalities, these results also agree with Lockett's results[2]. However, there is at least one point where an inconsistency occurs. Lockett indicates that if interface waves exist, they occur for incident angles greater than some critical angle. It was shown here that, except for the above mentioned special cases, interface waves occur for discrete incident angles, i.e., the reflected or transmitted ray moves away from the interface if the incident angle is increased beyond the critical angle.

The solution presented here includes all cases of homogeneous and isotropic linearly viscoelastic materials including the limiting elastic case. Once the complex angles $\zeta_{\ell mn}$ are determined, the results from Section II can be applied to determine the field for a fixed frequency ω. If pulse propagation is of interest, then a Fourier synthesis may be used to solve the transient problem.

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LIST OF FIGURES

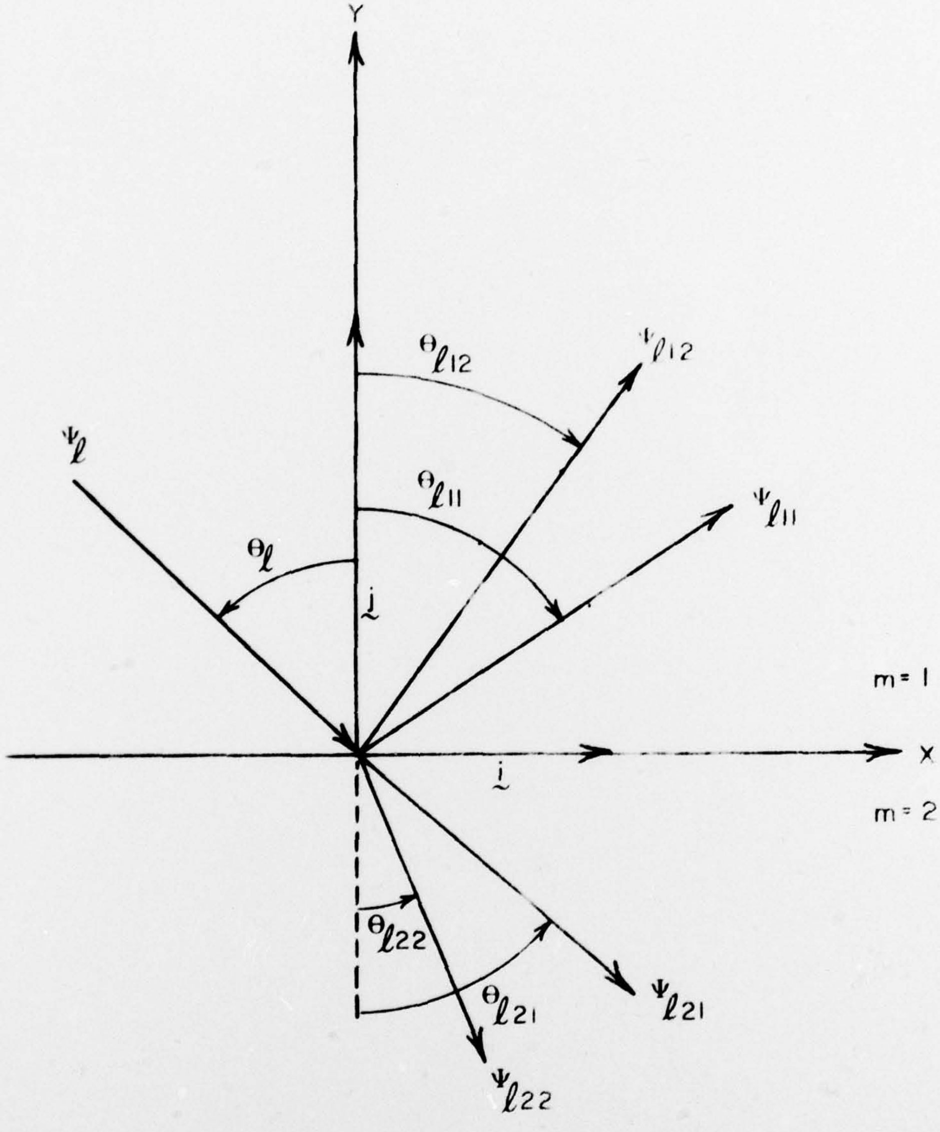

FIGURE 1 - REFLECTION AND REFRACTION PHENOMENA AT THE INTERFACE Y = 0

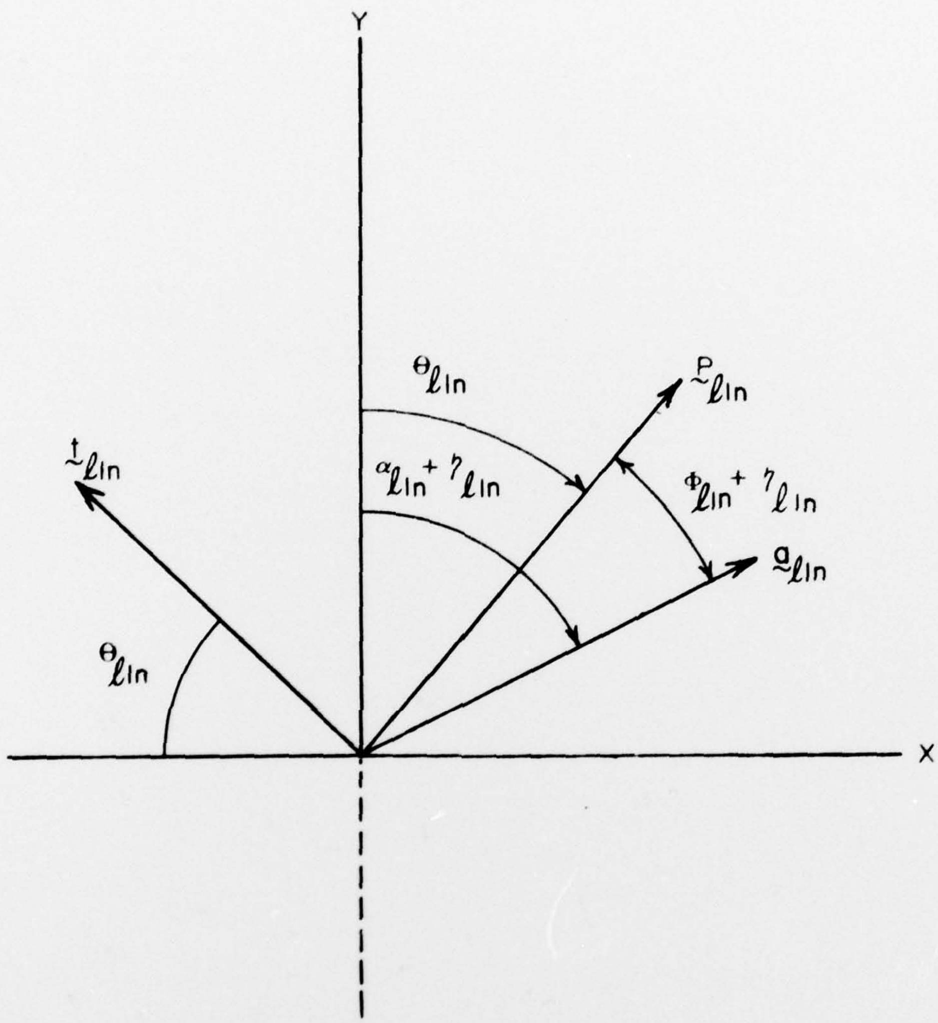

FIGURE 2 – PROPAGATION AND DAMPING DIRECTIONS OF REFLECTED WAVE $\Psi_{\ell 1n}$